Hybrid Algorithm for Route Design on Bus Rapid Transit Systems

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Abstract

In recent years, well-designed bus rapid transit (BRT) systems have become a real alternative to more expensive rail-based public transportation systems around the world. However, once the BRT system is operational, its success often depends on the routes offered to passengers. Thus, the Bus Rapid Transit Route Design Problem (BRTRDP) is the problem of finding a set of routes and frequencies that minimizes the operational and passenger costs (travel time) while simultaneously satisfying the system's technical constraints, such as meeting the demands for trips, bus frequencies, and lane capacities. To address this problem, we propose a mathematical formulation of the BRTRDP as a mixed-integer program (MIP) with an underlying network structure. However, because of the vast number of routes, solving the MIP via branch and bound is out of reach for most practical instances. Hence, we propose a decomposition strategy that, given a certain set of routes, decouples the route selection decisions from the BRT system performance evaluation. The latter evaluation is done by solving a linear optimization problem using a column generation scheme. We embedded this decomposition strategy in a hybrid genetic algorithm (HGA) and tested it in 14 instances ranging from 5 to 40 stations with different BRT system topologies. The results show that in 8 out of 14 problems, the HGA was able to obtain a solution that is provably optimal within 0.20%. Additionally, in 4 out of 14 instances, HGA obtained the optimal solution.

Key words: bus rapid transit systems; public transit network design; bus routing; urban logistics; matheuristics; genetic algorithms.

1 Introduction

A Bus Rapid Transit (BRT) system is a flexible, rubber-tired, high-capacity, low-cost public transit solution that is a competitive alternative to more expensive rail-based systems. Most specifically, BRT systems combine specialized buses, dedicated lanes, stations, off-vehicle fare collection, and intelligent transportation systems (ITS) into an integrated system with a strong identity under a unique image Danaher et al. (2007), Levinson et al. (2003).

The re-emergence of BRT is a worldwide initiative, with cities ranging from small to megasized adopting such systems. Just to name a few, in North America, BRT systems are in use in Los Angeles (US), Boston (US), and Ottawa (Canada); in Europe, in Leeds (UK) and Rouen (France); in Australia, in Sydney and Adelaide; and in South America, in Quito (Ecuador), Sao Paulo (Brazil), and Bogotá (Colombia). Indeed, a recent study by Hidalgo and Gutiérrez (2013) reports the application of 120 BRT systems around the world, covering in total more than 4,300 km in bus lanes, and serving about 28 million passengers per day.

One of the most highly recognized BRT implementations among transportation planners is TransMilenio TransMilenio (2013), Weinstock et al. (2011), which serves Bogotá, a city with 7,000,000 inhabitants citymayors.com (2013). By September of 2012, the system comprised an 87-km network of exclusive lanes, 115 stations, 1,392 articulated and bi-articulated buses (with 160- and 260-passenger capacity, respectively), and 90 routes. Already carrying over 1.5 million passengers per day by late 2012, TransMilenio was moving more than 198,000 passengers per peak hour by 2012, a volume normally associated with heavy rail transit modes. In addition, TransMilenio had increased average public transit travel speeds from 15 km/h to 27 km/h Cain et al. (2006), TransMilenio (2013). The great success of this BRT system has inspired other cities in Colombia and around the world to emulate the TransMilenio model Cain et al. (2006). In Colombia alone, the same model has been implemented by small-scale BRT systems in Barranquilla (TransMetro), Bucaramanga (MetroLínea), Cali (Mío), Cartagena (TransCaribe), Medellín (MetroPlús), and Pereira (MegaBus).

Nonetheless, despite the overwhelming success of BRT systems like TransMilenio, once in operation, they are subject to public complaints such as overcrowding and long wait times Cain et al. (2006), Cámara de Comercio de Bogotá (2010). In fact, a former TransMilenio operations manager claimed back in 2003 that the expected demand and passenger behavior of any BRT system in the planning stage often deviates from the system when it is in place and fully operational McAllister (2003). This mismatch between the planned and operational phases requires a thorough revision of the planned routes using a real origin-destination (OD) trip distribution matrix.

Thus, even though this research focuses on designing (or redesigning) BRT system routes, planning and operating any public transportation system raises a sequence of interrelated problems that must be tackled Ceder and Wilson (1986), Farahani et al. (2013), Guihaire and Hao (2008), Kepaptsoglou and Karlaftis (2009), Schöbel (2012). At the highest level of the hierarchy, the network design problem deals with deciding the physical layout of the transportation system. Laporte et al. (2002) solved the problem of deciding the best locations for the stations in a predesigned alignment (line) of a rapid transit system. Their approach is based on a longest path algorithm and a careful estimation of the catchment area. Likewise, Bruno et al. (2002) modeled the problem of locating a rapid transit alignment by maximizing the covered population subject to interstation spacing constraints. To solve the mathematical formulation, they proposed a two-phase algorithm that (1) constructs the alignment and (2) improves it. Even though designed for the single-alignment problem, their method could be used as a building block for designing rapid transit systems with multiple alignments. As a methodology for integrating the station location problem with that of connecting stations through a small number of alignments, Laporte et al. (2007) proposed an integer program that includes the construction cost as a constraint, which they illustrated using small six- and nine-node networks. Marín (2007) extended this model by allowing it to choose a variable number of alignments without predetermined origins and destinations. Unfortunately, however, their solution does not seem to scale well and was only tested on instances of up to nine nodes. In a subsequent study, Marín and Jaramillo (2008) incorporated an accelerated Benders decomposition technique and were able to solve instances of up to 24 nodes. More recently, Laporte et al. (2011) introduced a mixed integer optimization problem aimed to design robust transportation systems. The objective of their approach is to guarantee the existence of useful and fast routes even in the event of arc failures. They tested their approach by designing the routes for a nine-station system.

After having designed the physical network (e.g., station locations and alignments), the highest impact problem is that of *route design*. In the top-down approach of Ceder and Israeli (1992), the solution of this problem (i.e., the determination of a good set of bus routes) has a tremendous impact on subsequent tactical problems like bus frequency determination, timetabling, and bus and personnel allocation. These tactical problems, influenced by the designed routes and solved on a daily basis, have a direct impact on public opinion about the BRT system and the financial structure of the bus operators. Very often, cities adopting a BRT system lack sufficient room for expansion or simply find enlarging the network cost prohibitive, which not only makes the route design problem even more relevant but may leave it as the only alternative for improving BRT performance.

Because the re-emergence of BRT systems is a relatively recent endeavor, most extant literature concentrates on the more classic problem of routing buses that move freely within the road network while sharing it with other modes of transportation (for a review of the bus routing problem prior to 1990, see Odoni et al. (1994) and Chua (1984)). In that sense, the classic bus routing problem is to define which path on the road network enables buses to best serve passenger demand. On the other hand, because the buses of BRT systems like TransMilenio run on dedicated lanes (bus corridors), the bus paths are a predefined part of the physical infrastructure and the route design problem is that of selecting a subset of stations along the bus corridors at which the buses should stop.

Nonetheless, despite the differences between the classic bus routing on road networks and the route design of BRT systems, the bus routing literature does offer relevant information and solution techniques. Early approaches to solving the bus routing problem are based on constructive processes that basically assemble routes by connecting previously built fragments via the shortest paths. One of the first techniques following this approach is the skeleton method proposed by Silman et al. (1974), which generally starts by selecting a couple of terminal nodes in the city outskirts and then progressively creates a bus network by inserting intermediate stations based on passenger demand. Finally, it generates bus routes by connecting shorter sections (skeletons) found by solving shortest path problems between the intermediate stations.

Other researchers have solved the bus route design problem following a two-phase approach that first builds a set of routes able to operate the system and then selects the final routes using different heuristics. For example, Mandl (1980) constructed a set of candidate routes by solving a shortest path problem for each pair of nodes and then created routes from the pool of candidate paths by merging those with the largest number of nodes. He included unserved nodes by inserting them into the best position in terms of traveling time in a previously created route.

More recently, Baaj and Mahmassani (1991) have presented a three-component method that simultaneously solves the bus route design and bus dispatching frequency determination problems. The first component generates sets of routes; the second one uses a tool named Transit Routes Analyst (TRUST) Baaj and Mahmassani (1990) to compute route frequencies; and the third and last component uses TRUST to improve the previously generated routes. A sequel work by Baaj and Mahmassani (1995) revised the first component of the method, integrating a hybrid heuristic focused on generating the initial set of routes. The hybrid algorithm constructs skeletons by selecting high-demand node pairs and connecting them by –shortest or slightly longer– paths with different nodal compositions. Finally, the skeletons are expanded to routes based on four node insertion strategies that involve different performance measures.

Because of the problem's large scale, few studies have attempted mathematical programming techniques for designing routes. For example, Borndörfer et al. (2007) proposed a method based on column generation, in which they simultaneously define the routes and determine their frequencies. Nonetheless, their work focuses on solving the route design problem for a multimodal transportation system. A restrictive assumption in this work is the fact that transfers between lines are ignored because they greatly increase the complexity of the model. As stated by Borndörfer et al., handling these transfers fosters degeneracy and it remains unclear whether the resulting model remains tractable for practical purposes. Leiva et al. (2010) introduced a mixed-integer nonlinear program that considers line transfers and bus capacities. Because of the difficulty of solving the proposed formulation, the authors relaxed the nonlinear constraints and embed the resulting formulation within a row generation scheme. To test their approach, they solved the route design problem for a single-corridor BRT system with 10 stations. More recently, Feillet et al. (2010) presented a mathematical formulation for the BRT route design problem allowing route transfers. They proposed a simultaneous column and cut generation scheme, where the routes are systematically generated at each iteration. The authors were able to solve the BRT route design problem for single-corridor systems of up to to 19 stations.

Lately, some researchers have explored the use of metaheuristics for the bus route design problem. For example, Pattnaik et al. (1998) used genetic algorithms (GAs) to simultaneously solve the route design and frequency problems. First, after using shortest path solutions between each pair of nodes to produce an initial set of candidate routes (cf. Mandl 1980), they use a genetic search to find the best possible route collection for operating the system. The information of candidate routes is encoded within each individual of the genetic algorithm using a binary list whose routes are selected from those built using the shortest path problem. Chakroborty (2003), in contrast, used GAs to maximize the number of passengers moved by the system. However, unless an explicit cost or time objective is given, the proposed solutions tend to have more stops in every route, causing substantial delays for the passengers. Finally, using a similar approach, Cipriani et al. (2012) attempted to minimize a weighted sum of the operator's costs, the users' costs, and an additional penalty related to unsatisfied demands.

More recently, some researchers have used simulated annealing (SA) to solve the bus route design problem. For example, Fan and Machemehl (2006) proposed an SA with an objective function defined by the sum of the operational costs and an approximation of user costs associated with the time spent in the system. Alternatively, Zhao and Zeng (2006) proposed an objective function that minimizes total passenger transfers. Even though this objective seems reasonable, the proposed solutions tend to create routes with many stops that translate into delays. Although, it is possible to show that in some cases it is useful to use transfers to minimize passengers' total travel time. Finally, Fan and Mumford (2010) used hill-climbing and SA to design routes for ordinary bus systems by first creating an initial solution by solving shortest path problems and then running a local search improvement of the route set. A singular feature of this work was the inclusion of the number of transfers in the objective function together with the travel time.

For additional information regarding the route design and other related problems, see the comprehensive surveys by Farahani et al. (2013), Guihaire and Hao (2008), Kepaptsoglou and Karlaftis (2009), and Schöbel (2012).

Above all, most of the work in the literature on urban transit route design was originally formulated not for BRT systems but for the bus routing problem in road networks Ceder and Israeli (1992), Odoni et al. (1994), Chua (1984), Silman et al. (1974), Mandl (1980), Baaj and Mahmassani (1990, 1995), Pattnaik et al. (1998), Fan and Machemehl (2006) and unfortunately, it is not clear how to extend this methodologies to work on BRT systems. As a result, the route design problem for a BRT system remains unsolved.

To solve the route design problem in a BRT system, we propose a mixed-integer program (MIP) with an underlying network structure. Because of the vast number of possible routes, finding an exact solution via branch-and-bound is a very difficult endeavor for practical size instances; thus, we propose a decomposition strategy that, given a certain set of routes, decouples the route selection decisions from the BRT system performance evaluation. To carry out the latter evaluation, we use a large-scale linear programming technique that takes advantage of the underlying network structure to reduce the computational time needed by conventional optimization solvers to evaluate the performance of any given solution. To illustrate the decomposition scheme, we present a hybrid genetic algorithm (HGA) in which each solution is encoded in a binary genotype with multiple fragments, representing a set of routes able to operate the BRT system. The HGA then, uses the proposed BRT system performance evaluation as the fitness function. Additionally, we also propose a random solution generator based on a minimum cost network flow problem that is used to generate the initial population of the genetic algorithm. We finally used the genetic algorithm to solve 14 instances ranging from 6 to 40 stations with different BRT system topologies.

It is important to emphasize that despite the good results, the choice of the metaheuristic is somewhat arbitrary and should not overshadow the main contribution of this paper. In other words, the elements proposed in this paper (i.e., the decomposition strategy, the solution encoding, and the solution generator) are tested using a hybrid genetic algorithm, but they are general enough to be embedded in another optimization technique (heuristic or exact).

The remainder of this paper is organized as follows. Section 2 presents a description of the route design problem for a BRT system and puts forward the assumptions and data requirements for the problem, whereas §3 presents a network-oriented model of the BRT route design problem. Section 4 describes a decomposition strategy for solving it. Section 5 briefly discusses an extension of the network formulation to work on BRT systems with asymmetric routes, and §6 illustrates the proposed decomposition strategy embedded in a genetic algorithm. Section 7 illustrates the proposed approach on a set of computational experiments adapted from the literature and from our own experience. The hybrid approach is validated and compared against mixed-integer programming formulations solved with commercial optimizers. Finally, §8 concludes and outlines research currently underway.

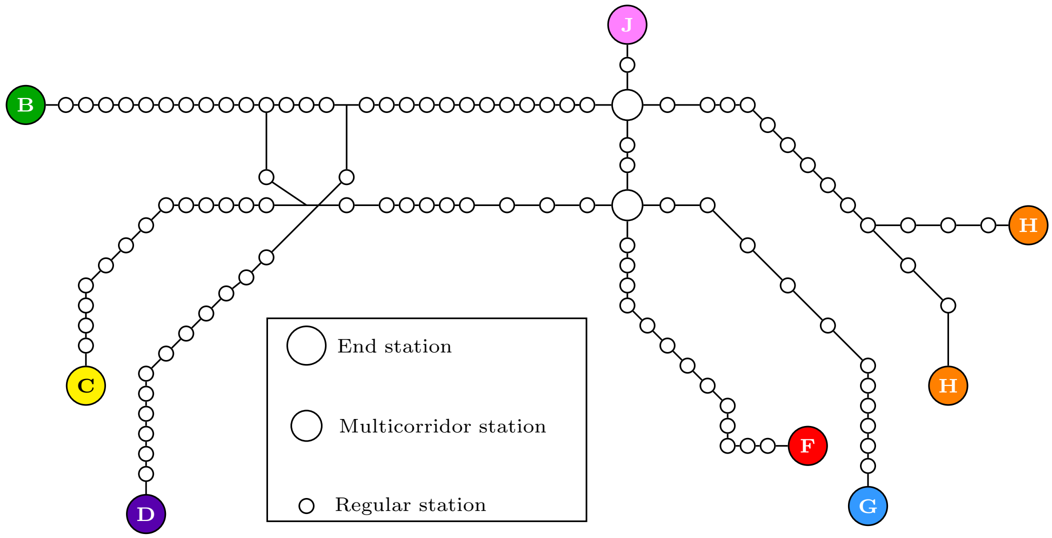
2 The BRT Route Design Problem

In BRT systems, following the globally recognized TransMilenio model, passengers pay a single fare at the gate of the station. Then, as in rail-based systems, they walk up a ramp toward a doorway at which they wait for a given route (bus). Once inside the system, passengers can travel between any pair of stations without leaving the network; however, to reach their destination, they may need to transfer to another bus following a different route. Bus transfers occur at intermediate stations where passengers wait for the next bus.

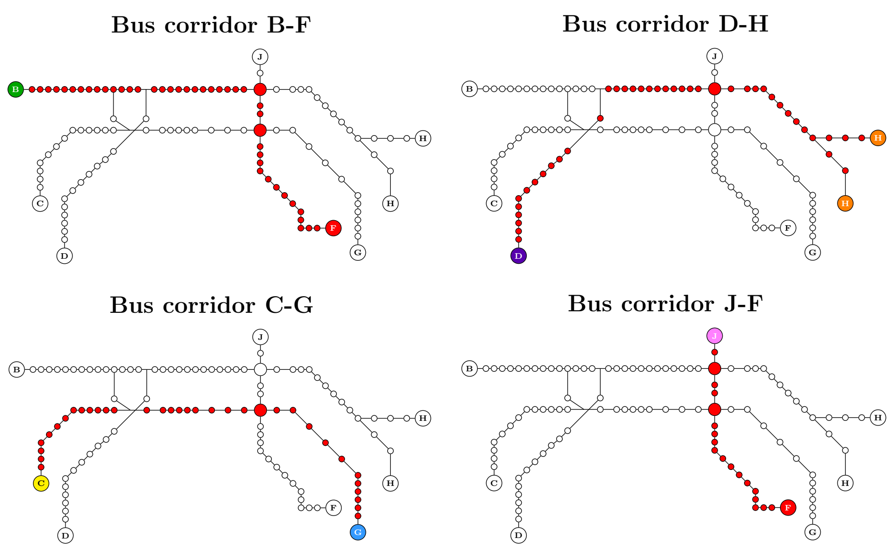
The stations, each associated with at least one bus corridor, are the only places in the BRT system where buses stop and passengers get on and off. Conceptually, a bus corridor is a series of physically connected bus lanes holding a set of adjacent stations. Routes are designed for each bus corridor, and conversely, every bus corridor is assigned a group of routes. One very important feature of BRT systems that follow the TransMilenio model is that buses are allowed to pass other buses stopped at stations, giving rise to *express routes* (i.e., those having few stops). To illustrate, Figure 1(a) shows the TransMilenio BRT system. It should be noted that one station can be shared by different bus corridors and are therefore *multicorridor* stations. Four such bus corridors in the TransMilenio are illustrated in Figure 1(b), while Figure 1(c) graphically represents routes J70 and J72 along the bus corridor B-J whose two end stations are B and J. Figure 1 clearly illustrates how the location of stops differs between both routes. Whereas route J72 stops at 16 stations widely spread along the bus corridor, route J70 only stops at 10 stations and by design skips stations located in the middle segment of the bus corridor. Hence, the latter has the particular structure of an express route.

The BRTRDP involves finding a manageable set of routes and frequencies that minimizes the operational and passengers costs while simultaneously satisfying system technical constraints –coping with the OD matrix (trip distribution), fleet size, and lane capacities– enforced so that the set of routes can satisfy demand without overcrowding the network (stations, buses, and lanes). Because of cultural issues and managerial efficiency, it is desirable to operate the system with a limited number of routes.

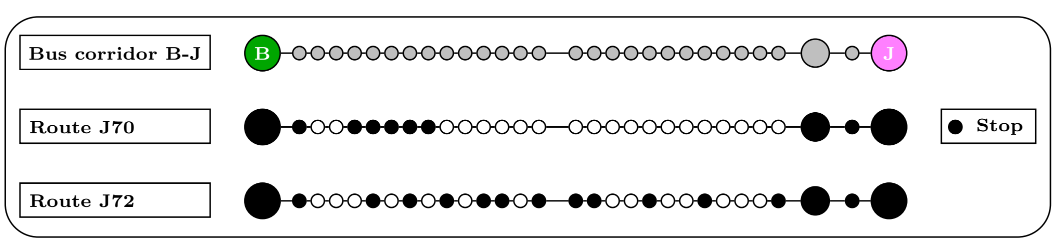
Formal expression of the BRTRDP, however, requires prior understanding of the following



(a) TransMilenio system



(b) Examples of four corridors



(c) Graphical representation of routes J70 and J72 running on bus corridor B-J

Figure 1: TransMilenio BRT system operating in Bogotá (as of June, 2010)

considerations, data requirements, and assumptions related to different aspects of the BRT system:

1. ***Solution Design.*** The passengers get used to the system, so it is not recommended to change the routes very often. Routes are redesigned only when new stations or corridors are included, or when the demand has suffered a significant change. Thus, it is not crucial for this method to be particularly fast.

2. ***Time horizon.*** The ideal time horizon should be decided based on how much the demand varies throughout the day. Given wide variation, the problem should be solved for peak hours because they represent a bottleneck during which system performance most strongly influences public opinion. The set of routes found for peak hours should thus also be valid for off-peak hours, but with lower frequencies.

3. ***Origin-Destination (OD) matrix.*** From among the many possible techniques for collecting and forecasting travel demand between stations, we draw on the two proposed by Balcombe et al. (2004): surveying system users and collecting information on ticket sales (the latter is particularly suitable when redesigning the routes). These data translate readily into demand for the public transit system and can be represented by an OD matrix in which each element contains the number of passengers willing to travel from one system station to another during a given time horizon. According to Borndörfer et al. (2007), the OD matrix is the simplest, standard, and most convenient way to estimate demand in a public transit system study. However, the quality of the solution of any model that relies on OD matrices significantly depends on the accuracy of the data. Different techniques for the estimation and calibration of the OD matrix can be found in Cascetta (2009) and Ortúzar and Willumsen (1994).

4. ***Route symmetry.*** For the sake of simplicity, this paper assumes that all routes are symmetric, meaning that the route stops at the same stations in both directions. Pragmatically, some TransMilenio-type BRT systems prefer this (symmetric) route structure because it makes the system easy to use. However, at the end of section §5 we provide information on how to extend the proposed technique to work on systems with asymmetric routes.

5. ***Passenger assignment.*** One critical issue while measuring the performance of a BRT system is to estimate the preferred travel paths of the passengers and the flow through those paths Desaulniers and Hickman (2007). Finding those paths and flows is particularly problematic during the planning stage because there is no way to validate against historical data. Thus, it is often the case to rely in the general assumption that all the passengers make their travel decisions based on a common objective function Correa et al. (2004), Desaulniers and Hickman (2007). This objective function is frequently addressed in the literature as the minimization of the total travel time Silman et al. (1974), Ceder and Wilson (1986), Borndörfer et al. (2007) or as the minimization of a generalized cost function, that is, a weighted sum of different components such as the

travel time, the waiting time, the transfer time, and the number of transfers Ceder and Israeli (1992), Baaj and Mahmassani (1995), Fan and Machemehl (2006).

Earlier work on passenger assignment was based on the all-or-nothing Silman et al. (1974), Mandl (1980) or common-lines Baaj and Mahmassani (1990) techniques. In both cases, the paths and frequencies are fixed, usually as the result of a previous stage (shortest path calculation). The main difference between both approaches is that while in the first, all the passengers of a given OD pair are assigned to one path (i.e., the shortest path); in the second approach, the flows are split between common paths according to the bus frequency (i.e., paths sharing the same cost).

More recent work on passenger assignment freely assigns the users without pre-defined paths Borndörfer et al. (2007). Following this approach, we use a mathematical formulation that simultaneously determines the passenger flow along with the frequency determination. The advantage of this approach is that, not only the passengers are optimally assigned according to their objective function, but also the model can adjust the frequencies in favor of the more congested routes. Nevertheless, to avoid nonlinearities in the objective function, we do not consider the effect of the congestion in the stations, passenger queues or bus bunching in the travel decisions of the passengers.

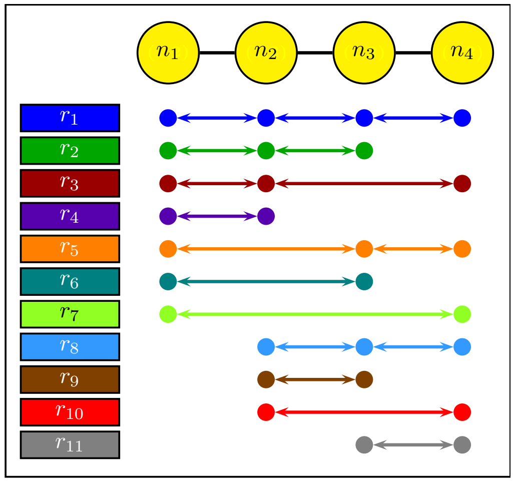
6. ***Multiobjective structure.*** Aside from the passengers' perspective, the operational cost is another key driver to consider while planning and operating a transportation system Desaulniers and Hickman (2007). The operational cost is often described in the literature as a function of the route length, the route frequency, and the cost related to the use of the infrastructure Borndörfer et al. (2007), Fan and Machemehl (2006). This multiobjective structure adds complexity to the problem because it often requires a-priori articulation of the preference between both objectives, as well as the proper estimation of the relative importance of the competing elements within each objective. In this work, as in most of the extant literature Baaj and Mahmassani (1991), Fan and Mumford (2010), Borndörfer et al. (2007), Fan and Machemehl (2006), we tackle the multiobjective structure under a classical weighted-sum approach. The proposed objective function is fully described in §3, but the question of how to calibrate those weights is out of the scope of this paper.

7. ***Route transfers.*** Not only are route transfers allowed, they are sometimes the only feasible way to move passengers between a given pair of stations. Hence, allowing transfers in a BRT system helps decrease the number of stops in a route and decreases the number of routes needed to operate the system. Nonetheless, transfers are not instantaneous; rather, a transfer time includes the time the passenger needs to walk inside the transfer station (between doors on a given ramp) and the wait time for the next bus.

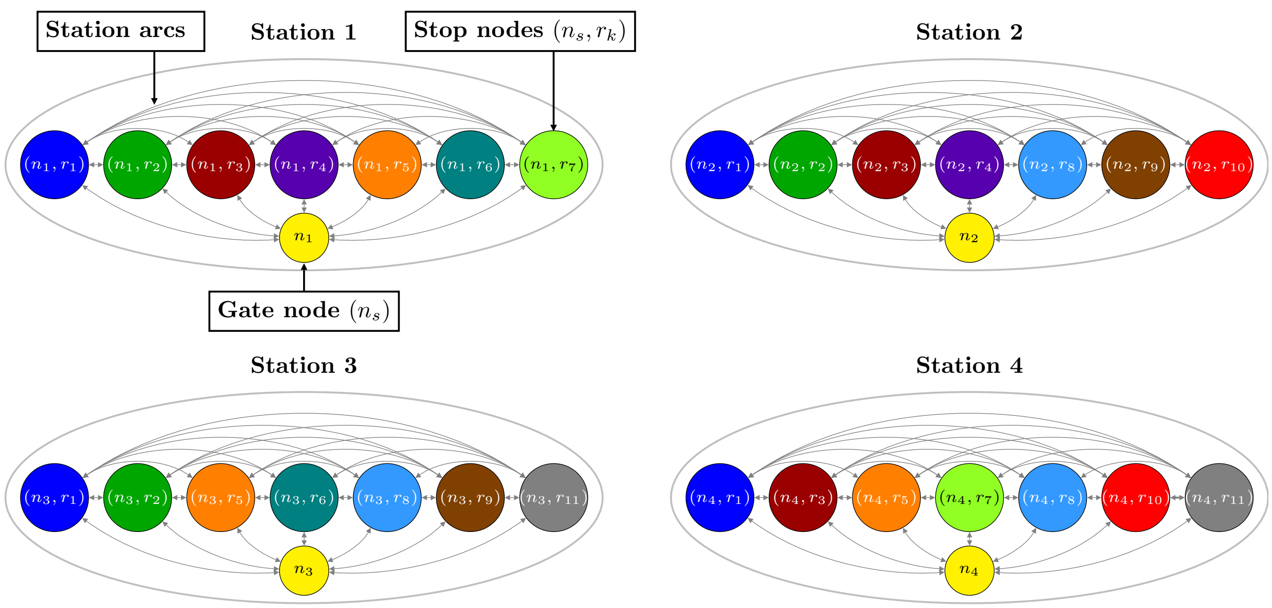
3 Network Model of the BRTRDP

The mathematical formulation of the BRTRDP is based on a network representation of the BRT system. Consider a directed graph $G(\mathcal{N}, \mathcal{A})$, where $\mathcal{N}$ represents the set of nodes and $\mathcal{A}$ the set of directed arcs, then let $\mathcal{S}$ be the set of system stations and $\mathcal{R}$ the set of all feasible routes able to operate the system. There is a node for every station $n_s \in \mathcal{S}$ representing the gate to station n_s; these nodes are denominated *gate nodes*. There is a node for every pair (n_s, r_k), if the route r_k stops at station n_s. These nodes represent the boarding doors on the station n_s ramp at which passengers wait for a bus serving route r_k; these nodes are denominated *stop nodes*. Let $s(n_i)$ be the station associated with the stop node n_i and $\mathcal{N}(r_k)$ be the set comprised of the stop nodes of route r_k. Hence, $\mathcal{N} = (\bigcup_{r_k \in \mathcal{R}} \mathcal{N}(r_k)) \cup \mathcal{S}$. Within a station n_s there is an arc between every pair of stop nodes and an arc between the gate node and every stop node. These arcs are denominated *station arcs*, are denoted $\mathcal{A}(n_s)$ for station n_s and model the movement of the passengers inside the station. For every route r_k, there is a set of arcs $\mathcal{A}(r_k)$ connecting the stop nodes $\mathcal{N}(r_k)$ following the sequence defined by the route. These arcs are denominated *route arcs* and model the physical movement of a bus between stops through busways based on the sequence defined by the route. Then, $\mathcal{A} = (\bigcup_{r_k \in \mathcal{R}} \mathcal{A}(r_k)) \cup (\bigcup_{n_s \in S} \mathcal{A}(n_s))$. Consider the network representation for a four-station BRT system with one bus corridor illustrated in Figure 2. Here, set $\mathcal{R} = \{r_1, r_2, r_3, r_4, r_5, r_6, r_7, r_8, r_9, r_{10}, r_{11}\}$ listed in Figure 2(a), which shows all possible routes for this BRT system, exclude the infeasible routes like nonstop or single-stop routes. Additionally, set $\mathcal{S} = \{n_1, n_2, n_3, n_4\}$ contains all gate nodes. In Figure 2(b), which details the subnetwork corresponding to the stations, because routes r_1 through r_7 stop at station n_1, the corresponding subnetwork contains the (n_s, r_k) nodes $n_{(n_1,r_1)}$, $n_{(n_1,r_2)}$, $n_{(n_1,r_3)}$, $n_{(n_1,r_4)}$, $n_{(n_1,r_5)}$, $n_{(n_1,r_6)}$, and $n_{(n_1,r_7)}$. Thus, for instance, $s(n_{(n_1,r_2)}) = n_1$. Figure 2(c) then shows the resulting BRT network after the expansion of the station subnetworks and addition of the route arcs. Note that the route arcs presented in the figure actually represent two directed arcs in both directions; for example, route r_{10} (red) only stops at stations n_2 and n_4, then, $\mathcal{N}(r_{10}) = \{n_{(n_2,r_{10})}, n_{(n_4,r_{10})}\}$ and $\mathcal{A}(r_{10}) = \{(n_{(n_2,r_{10})}, n_{(n_4,r_{10})}), (n_{(n_4,r_{10})}, n_{(n_2,r_{10})})\}$.

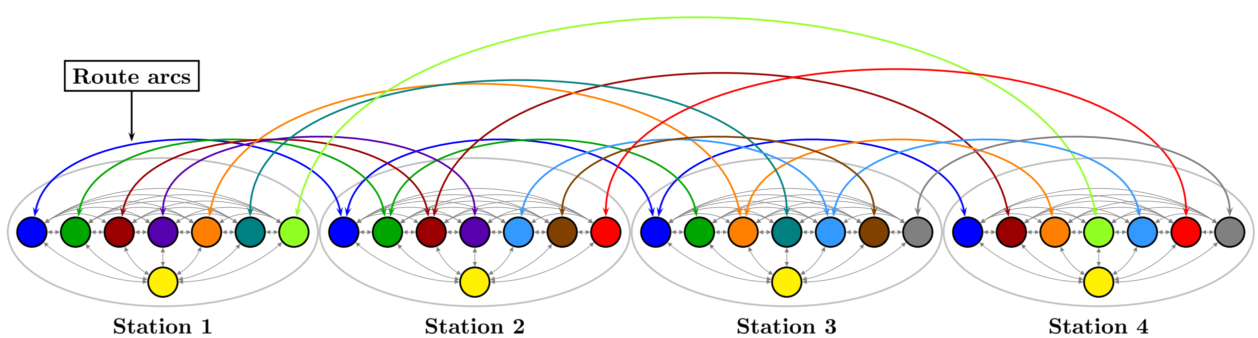
Two types of costs are considered: the passenger cost of using the BRT system and the BRT's operational costs. Let c_{ij} be the cost (e.g., time) that a passenger incurs when traversing arc (n_i, n_j). If this arc (n_i, n_j) is a station arc, then c_{ij} represents the walking time from the gate to a boarding door, from a boarding door to the gate, or the transfer time (the walking time between boarding doors plus the average wait) time for the next bus. If arc (n_i, n_j) is a route arc, then c_{ij} represents the time the bus takes to travel between two successive stop nodes; this time is assumed to be departure independent of the route, i.e., it only depends on the departure and arrival stations. We also denote this travel time between stations $s(n_i)$ and $s(n_j)$ by $c_{s(n_i)s(n_j)}$. Let g_k be a fixed cost of operating route r_k and h_k be the cost of allocating a bus to serve route r_k. Let b^{od} be the number of passengers willing to travel from station n_o to station n_d (i.e., the (o, d) element of the OD matrix). From this value, b_i^{od} is defined for every node n_i, where $b_o^{od} = b^{od}$, $b_d^{od} = -b^{od}$, and $b_i^{od} = 0$ otherwise. It should be noted that stop nodes act as transshipment nodes. Let Q be the number of buses available to



(a) Feasible routes



(b) Subnetworks representing each station



(c) Full network representation

Figure 2: Example of a four-station BRT system

serve the BRT system and u be the passenger capacity of each bus. For the sake of simplicity, we assume that the fleet of buses is homogeneous (i.e., all the buses have the same capacity). Let q_k be the maximum number of buses that could be allocated to route r_k and m be the maximum number of routes that the BRT operator is able to manage. Let x_{ij}^{od} be a decision variable that represents the flow through arc (n_i, n_j) of passengers willing to go from station n_o to station n_d. Let y_k be a decision variable that takes the value of 1 if route r_k is selected to operate the BRT system; it takes the value of 0 otherwise. Finally, let f_k be the number of buses allocated to serve route r_k (i.e., frequency). Then, the BRTRDP is as follows:

$$\min \qquad \alpha \left(\sum_{(n_i,n_j)\in\mathcal{A}} c_{ij} \sum_{n_o\in\mathcal{S}} \sum_{n_d\in\mathcal{S}} x_{ij}^{od} \right) + \beta \left(\sum_{r_k\in\mathcal{R}} (g_k y_k + h_k f_k) \right) \tag{1}$$

s.t.

$$\sum_{\{n_j:(n_i,n_j)\in\mathcal{A}\}} x_{ij}^{od} - \sum_{\{n_j:(n_j,n_i)\in\mathcal{A}\}} x_{ji}^{od} = b_i^{od}, \qquad n_i \in \mathcal{N}, n_o \in \mathcal{S}, n_d \in \mathcal{S}; \tag{2}$$

$$\sum_{n_o\in\mathcal{S}} \sum_{n_d\in\mathcal{S}} x_{ij}^{od} \le u f_k; \qquad r_k \in \mathcal{R}, (n_i, n_j) \in \mathcal{A}(r_k) \tag{3}$$

$$f_k \le y_k q_k, \qquad r_k \in \mathcal{R}; \tag{4}$$

$$\sum_{r_k\in\mathcal{R}} y_k \le m; \tag{5}$$

$$\sum_{r_k\in\mathcal{R}} f_k \le Q; \tag{6}$$

$$y_k \in \{0,1\}, \quad r_k \in \mathcal{R}; \tag{7}$$

$$f_k \ge 0, \qquad r_k \in \mathcal{R}; \tag{8}$$

$$x_{ij}^{od} \ge 0, \qquad n_o \in \mathcal{S}, n_d \in \mathcal{S}, (n_i, n_j) \in \mathcal{A}; \tag{9}$$

where the objective function (1) simultaneously minimizes the passenger travel cost and the operational cost of the BRT system. Note the existing compromise between both objectives, that is, minimizing the passenger travel cost would result in a larger number of routes and buses serving the BRT, which implies a deterioration on the revenue for the operator. For this reason, scalar weights α and β are used to combine both objectives under a classical weighted-sum approach (a-priori articulation of preferences). Balance constraints (2) guarantee that all passengers reach their destination. Capacity constraints (3) force the flow of passengers for every arc to be less than the combined capacity of the buses allocated to serve a given route. Constraints (4) ensure that if a route is selected to operate the system, the buses allocated to serve such route do not exceed the maximum allowed. Constraint (5) guarantees that the number of selected routes does not exceed a manageable limit set by the BRT system operator. Finally, constraint (6) bounds the number of buses.

Even though the model defined by (1–9) fully describes the BRTRDP, solving it is unfortunately a difficult task. The BRTRDP falls into a broader category of complex problems known as network design problems, comprised among others by the fixed-charge network design problem Magnanti and Wong (1984), the capacitated network design problem Balakrishnan et al. (1997), and the capacitated multicommodity network design problem Frangioni and Gendron (2009), all of them proven to be NP-hard.

We now outline how the capacitated multicommodity network design problem (CMND), is polynomially reducible to the BRTRDP. From the CMND formulation Ghamlouche et al. (2003) we can associate each arc (n_i, n_j) of its underlying network with a route r_k, thus we are able to rewrite variables y_{ij} as y_k for every arc (n_i, n_j) such that $\mathcal{A}(r_k) = \{(n_i, n_j)\}$. Then, after fixing the parameters $\alpha = 1$, $\beta = 1$, $g_k = 0$, $m = \infty$, and $Q = |\mathcal{A}|$; and eliminating the redundant constraints in the BRTRDP formulation (1–9), the resulting problem is indeed an instance of the CMND, proving that the BRTRDP is in fact an NP-hard problem. As a result, approaches to tackle the BRTRDP or any other network design problem rely on heuristic techniques to solve practical instances of reasonable size Ghamlouche et al. (2003).

Aside from its worst-case analysis, note that the size of the BRTRDP defined by (1–9) in terms of the number of variables and constraints is intimately tied to the set of routes $\mathcal{R}$. Moreover, due to the combinatorial structure of the problem the total number of routes is $O(2^{|\mathcal{S}|})$. Since for every route r_k the underlying network contains the sets of stop nodes $\mathcal{N}(r_k)$ and route arcs $\mathcal{A}(r_k)$, thus the number of constraints (2–6) and variables grows exponentially with the number of stations. This implies that in order to solve the proposed model defined by (1–9), it is necessary to solve a problem with an extremely large number of constraints and variables. Indeed, this number could be huge even for small instances (see § 7).

4 Decomposition Strategy for the BRTRDP

The BRTRDP defined by (1–9) is undoubtedly a hard large-scale optimization problem. Even small-sized instances are out of reach for commercial branch-and-bound based optimizers. However, if somehow the set of routes that serves the BRT system is fixed, the size of the network reduces dramatically and the resulting problem becomes well defined. In other words, the problem is reduced to the evaluation of the BRT system given that set of routes. Formally, the idea is to select a small collection of routes $\mathcal{R}' \subset \mathcal{R}$ ($|\mathcal{R}'| << |\mathcal{R}|$), fix the corresponding y_k variables to 1 for every route $r_k \in \mathcal{R}'$ (and $y_k \leftarrow 0$ for all $r_k \in \mathcal{R} \setminus \mathcal{R}'$), construct a graph $G'(\mathcal{N}', \mathcal{A}')$ by pruning nodes and arcs in G given $\mathcal{R}'$, and find the overall cost of the resulting BRT system. To evaluate the performance of the BRT system, given a set of routes represented by $\mathbf{y}$, we solve the following linear optimization problem:

$$f(\mathbf{y}) = \min \qquad \alpha \left(\sum_{(n_i,n_j) \in \mathcal{A}'} c_{ij} \sum_{n_o \in \mathcal{S}} \sum_{n_d \in \mathcal{S}} x_{ij}^{od} \right) + \beta \left(\sum_{r_k \in \mathcal{R}'} h_k f_k \right) \tag{10}$$

s.t.

$$\sum_{\{n_j:(n_i,n_j)\in\mathcal{A}'\}} x_{ij}^{od} - \sum_{\{n_j:(n_j,n_i)\in\mathcal{A}'\}} x_{ji}^{od} = b_i^{od}, \quad n_i \in \mathcal{N}', n_o \in \mathcal{S}, n_d \in \mathcal{S}; \tag{11}$$

$$\sum_{n_o\in\mathcal{S}}\sum_{n_d\in\mathcal{S}} x_{ij}^{od} \le uf_k, \quad r_k \in \mathcal{R}', (n_i,n_j) \in \mathcal{A}(r_k); \tag{12}$$

$$f_k \le q_k, \quad r_k \in \mathcal{R}'; \tag{13}$$

$$\sum_{r_k\in\mathcal{R}'} f_k \le Q; \tag{14}$$

$$x_{ij}^{od} \ge 0, \quad n_o \in \mathcal{S}, n_d \in \mathcal{S}, (i,j) \in \mathcal{A}'; \tag{15}$$

$$f_k \ge 0, \quad r_k \in \mathcal{R}'. \tag{16}$$

Note that, since most of the nodes and arcs were pruned when fixing variables $\mathbf{y}$, the model defined by (10–16) is significantly smaller than the BRTRDP formulation (1–9). Moreover, because all the variables in the model are continuous, this is in fact a linear optimization model, which theoretically can be solved in polynomial time. Additionally, note that this model closely resembles a multicommodity network flow problem (MCNF) in which each commodity represents passengers traveling from station n_o to station n_d. Despite the extra variables f_k, we can successfully take advantage of the MCNF structure, via the decomposition principle. The most important implication of this formulation is that, thanks to its structure, it is possible to evaluate in a relatively short time how well the BRT system will perform given the set of routes defined by $\mathbf{y}$.

To solve the problem defined by (10–16), we use a variant of the decomposition principle proposed by Tomlin (1966) originally conceived to solve MCNF problems. There are also other solution approaches suggested in the literature McBride (1998), Ahuja et al. (1993) that could be adapted to solve the problem. The decomposition principle allows the acceleration of the solution process by taking advantage of the problem's internal network structure. The decomposition principle splits the original problem in two, giving rise to a *master* and an *auxiliary* problem. In a given iteration, the auxiliary problem provides the master problem with possible paths, the master problem then receives these paths, builds a new basis, and returns the dual pricing to the auxiliary problem so it can declare optimality or construct other paths if necessary.

The master problem is defined as follows. Let $\mathcal{P}_{od}$ be the set of all possible paths that satisfy the balance constraints (11) for the pair of stations $(n_o, n_d) \in \mathcal{S} \times \mathcal{S}$. Let w_{ijp} be the flow of passengers through the arc $(n_i, n_j) \in \mathcal{A}'$, in the path $p \in \mathcal{P}_{od}$. Let λ_p^{od} be a multiplier associated with the p^{th} path $p \in \mathcal{P}_{od}$. Hence, the master problem is as follows:

$$f(\mathbf{y}) = \min \quad \alpha \left(\sum_{(n_i,n_j)\in\mathcal{A}'} c_{ij} \sum_{n_o\in\mathcal{S}}\sum_{n_d\in\mathcal{S}}\sum_{p\in\mathcal{P}_{od}} \lambda_p^{od} w_{ijp} \right) + \beta \left(\sum_{r_k\in\mathcal{R}'} h_k f_k \right) \tag{17}$$

s.t.

$$\sum_{n_o \in \mathcal{S}} \sum_{n_d \in \mathcal{S}} \sum_{p \in \mathcal{P}_{od}} \lambda_p^{od} w_{ijp} \leq u f_k, \quad r_k \in \mathcal{R}', (n_i, n_j) \in \mathcal{A}(r_k); \tag{18}$$

$$\sum_{r_k \in \mathcal{R}'} f_k \leq Q; \tag{19}$$

$$f_k \leq q_k, \quad r_k \in \mathcal{R}'; \tag{20}$$

$$\sum_{p \in \mathcal{P}_{od}} \lambda_p^{od} = 1, \quad n_o \in \mathcal{S}, n_d \in \mathcal{S}; \tag{21}$$

$$\lambda_p^{od} \geq 0, \quad n_o \in \mathcal{S}, n_d \in \mathcal{S}, p \in \mathcal{P}_{od}; \tag{22}$$

$$f_k \geq 0, \quad r_k \in \mathcal{R}'; \tag{23}$$

where the objective function of the master problem (17), as in the original problem, is to minimize the passenger travel cost and the operational cost of the BRT system; the inequalities (18) represent the arc capacity constraints; relations (20) ensure that the buses allocated to serve the routes do not exceed the maximum allowed; constraint (19) bounds the number of buses; and equations (21) represent the convexity constraints for every pair of stations (n_o, n_d). Note that the set of variables f_k in the master problem is fixed over the iterations, since it only depends on the set of routes $\mathcal{R}'$ that is being evaluated by the algorithm. On the other hand, the sets $\mathcal{P}_{od}$ grow with the paths built by the auxiliary problem at each iteration.

The auxiliary problem is separable with a diagonal block structure in which each block relates to the passenger flow between a pair of stations (n_o, n_d). Let π_{ij}^k be the dual variables of the capacity constraints of arc $(n_i, n_j) \in \mathcal{A}(r_k)$ (18) and γ^{od} be the dual variable associated with the convexity constraint (21) for the pair (n_o, n_d). The auxiliary problem (block) for the passengers (n_o, n_d) is defined by:

$$\min \quad \sum_{(n_i,n_j) \in \mathcal{A}'} \left(\alpha c_{ij} - \sum_{r_k \in \mathcal{R}'} \pi_{ij}^k \right) x_{ij}^{od} - \gamma^{od} \tag{24}$$

s.t.

$$\sum_{\{n_j:(n_i,n_j) \in \mathcal{A}'\}} x_{ij}^{od} - \sum_{\{n_j:(n_j,n_i) \in \mathcal{A}'\}} x_{ji}^{od} = b_i^{od}, \quad i \in \mathcal{N}'; \tag{25}$$

$$x_{ij}^{od} \geq 0, \quad (n_i, n_j) \in \mathcal{A}'; \tag{26}$$

where the set of constraints (25) guarantees that the offer and demand for passengers (n_o, n_d) is met in every node. Since there is no limit on passenger flow (n_o, n_d), this problem can be solved by pushing all the (n_o, n_d) flow through the shortest path. Hence, the solution to this problem can take advantage of a specialized algorithm.

The optimality conditions can be written as follows:

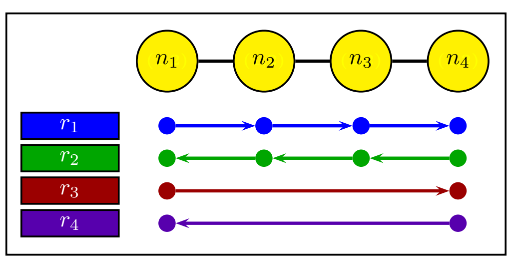
$$\sum_{(n_i,n_j) \in \mathcal{A}'} \left(\alpha c_{ij} - \sum_{r_k \in \mathcal{R}'} \pi_{ij}^k \right) x_{ij}^{od} \geq \gamma^{od}, \quad (n_o, n_d) \in \mathcal{S} \times \mathcal{S}. \tag{27}$$

If these conditions are met, it means that there are no more paths that decrease the overall cost, thus, the current solution in the master problem is optimal.

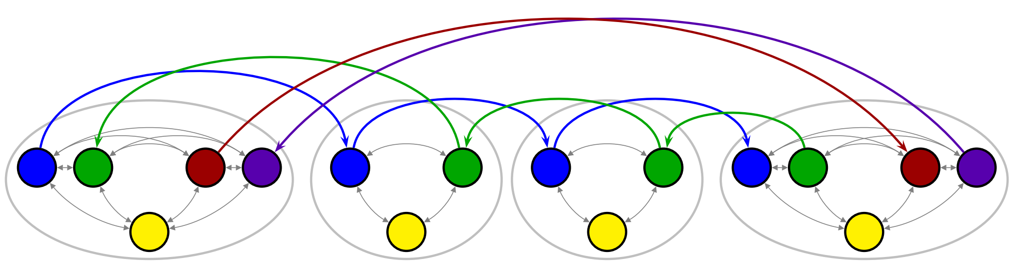
To solve the BRTRDP, the proposed performance evaluation tool defined by (10–16) can be embedded within an optimization framework working on a solution space comprised of the set of routes able to operate the BRT system. In Section § 6, we illustrate such an optimization scheme based on a genetic algorithm that uses the previous model derived in this section as a fitness function. Nonetheless, other effective search mechanisms in combinatorial optimization (e.g., a tabu search) can also take advantage of the same decomposition approach presented here.

5 Extension to the Network Formulation to Asymmetric Routes

The network formulation can be extended to work on BRT systems with asymmetric routes, that is, routes that do not stop at the same stations in both directions. To model asymmetric routes, first, for every route we define a traveling direction along the corridor. For example in Figure 2 the two possible directions are 1–4 and 4–1. The network is built as before, but every stop node is duplicated to account for each traveling direction. Additionally, for every route r_k, the set of route arcs $\mathcal{A}(r_k)$ is now comprised only by the arcs following the direction, that is, either (n_i, n_j) or (n_j, n_i), but not both. Note that under this approach, handling both directions doubles the number of candidate routes, thus affecting the size of the optimization model. Figure 3 shows an example of the network representation for the same four-station system used in Figure 2. For the sake of clarity, we display only four from the whole set of 22 set of candidate routes (the same 11 routes described in Figure 2, but in both directions). Note that even though routes r_1 and r_2 stop at the same stations, they travel in different directions.



(a) Sample of four asymmetric routes



(b) Network representation of the sampled routes

Figure 3: Example of a four-station BRT system with asymmetric routes

6 Hybrid Genetic Algorithm

Genetic algorithms Goldberg (1989) are stochastic search procedures inspired by the principles of natural selection. In any given generation (i.e., iteration), genetic operators combine and alter a population of solutions that undergoes an evaluation and selection process in which the fitter individuals (i.e., solutions) are more likely to survive. To improve the convergence of genetic algorithms, some researchers have explored hybrid genetic algorithms, which, in a strict sense, are those composed of simple algorithms Black (2004). However, because there appears to be no general agreement on the use of the term *hybrid*, we use it here in the sense of decoding a chromosome (solution encoding) and evaluating its fitness by means of an exact algorithm. Even though this approach might seem related to what some authors call hybrid genetic algorithms Whitley (1995), Cheng et al. (1999), other researchers prefer the term memetic algorithms to emphasize the use of (local) search procedures to intensify the genetic search Moscato (1999). In our view, the proposed approach can also be seen as a matheuristic, a term that has been recently coined to emphasize the cross-fertilization (hybridization) between mathematical programming and metaheuristics Caserta and Voß (2010). Hence, this section describes the key components of the proposed hybrid genetic algorithm, hereafter called HGA.

6.1 Solution encoding (genotype).

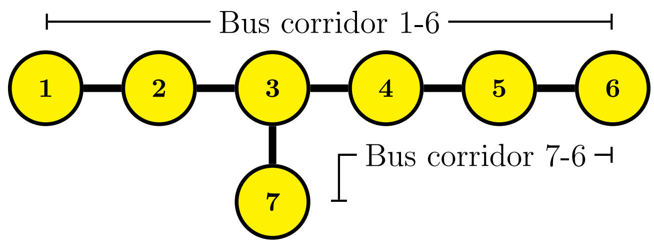
Each individual represents a collection of maximum m routes able to operate the system. The routes allocated to a given bus corridor are coded in a list of binary arrays with each array representing a route in which each bit represents a station in the bus corridor. A bit with a value of 1 indicates that the route stops at the corresponding station, whereas a value of 0 indicates that the route does not stop. It should be noted that in the illustration of a 7-station BRT system with two bus corridors shown in Figure 4, the number of routes assigned to each bus corridor could differ between individuals. In fact, such solution encoding allows the possibility that a bus corridor might even end up with no routes allocated after a number of iterations.

6.2 Fitness function.

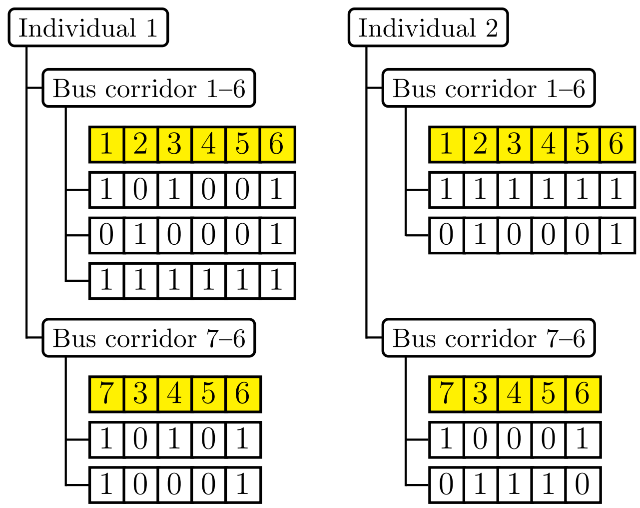
The key feature of this proposed genotype is that, because the binary variables $\mathbf{y}$ described in §3 are coded in the individual, the fitness of an individual can be obtained by simply solving the problem defined by (10–16) instead of the complex mixed-integer program (MIP) described by (1–9). For this purpose, we used the decomposition strategy presented in Section §4.

6.3 Initial population.

A set of routes $\mathcal{R}'$ with cardinality $m' \leq m$ is randomly assigned to every individual in the initial population. Each route $r_k \in \mathcal{R}'$ is associated with a bus corridor $c \in \mathcal{C}$, where $\mathcal{C}$ is the set of bus corridors in the system. To generate routes for a new individual, we define a



(a) System description



(b) Genotype

Figure 4: A 7-station example

minimum cost flow problem (MCF) on a directed graph, where route stops are the units that flow through the network.

Let the fraction $\rho(c)$ of routes assigned to bus corridor c be proportional to the number of stations $s(c)$ in the bus corridor as defined by $\rho(c) \triangleq s(c)/\sum_{c' \in \mathcal{C}} s(c')$. Let $\delta(r_k)$ be a function that returns the bus corridor of a given route r_k. For example, if route $\overline{r}_k$ is assigned to bus corridor $\overline{c}$, then $\delta(\overline{r_k}) = \overline{c}$. There is a *route node* for every $r_k \in \mathcal{R}'$ (m' route nodes in total); a *station node* for every $n_s \in \mathcal{S}$; and two nodes f and d denominated *source* and *sink* nodes, respectively. The source node pushes all the units (route stops) through the network, while the sink node absorbs them. There is an arc between the source node f an every route node r_k, all which are denominated *corridor-route arcs* and grouped in set $\mathcal{A}(f)$. Likewise, there is an arc between each route node r_k and every station node in bus corridor $c = \delta(r_k)$, denominated *route-stop arcs* and further grouped in set $\mathcal{A}(c)$ and if a unit flows through this arc, it means that route r_k stops at station n_s. Finally, there is an arc between every station node n_s and the sink node d, all denominated *station-stop arcs* and grouped in set $\mathcal{A}(d)$. Figure 5 presents an example for the 7-station BRT system previously presented in Figure 4(a).

Let $l(r_k)$ and $u(r_k)$ be the lower and upper bounds associated with the flow on arc $(f, r_k) \in \mathcal{A}(f)$, they represent the minimum and maximum number of stops of route r_k. To be feasible, the number of stops in a route must fall between two and the number of stations along the

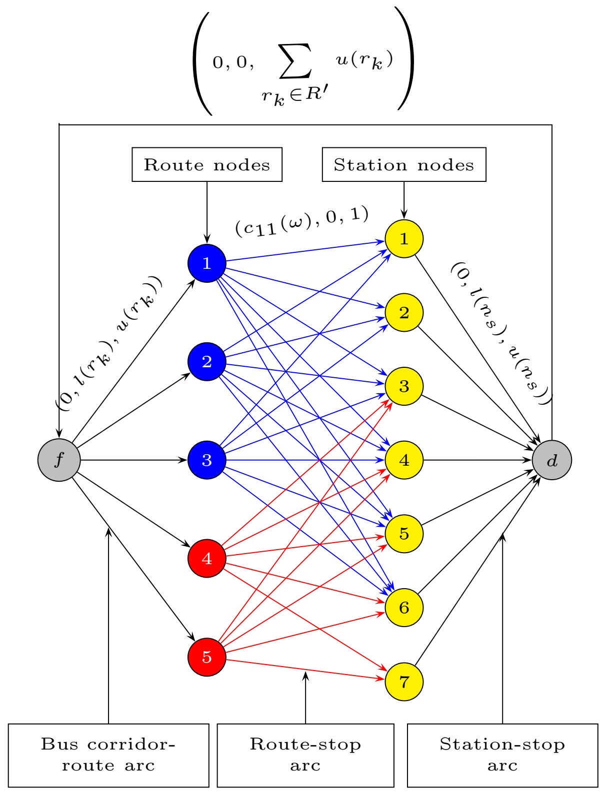


Figure 5: Example of the underlying minimum cost flow network for the initial population ($m' = 5$)

bus corridor, meaning $l(r_k) \triangleq 2$ and $u(r_k) \triangleq s(\delta(r_k))$. The cost associated with the arcs in $\mathcal{A}(f)$ is zero. For every $(r_k, n_s) \in \mathcal{A}(\delta(r_k))$, let $c_{ks}(\omega)$ be the cost of allocating a stop at station n_s in route r_k, where ω is a random effect that is responsible for the diversity of the newly generated individuals and might therefore take negative values and be different for each $(r_k, n_s) \in \mathcal{A}(\delta(r_k))$. Every arc $(r_k, n_s) \in \mathcal{A}(\delta(r_k))$ has lower and upper bounds of 0 and 1, respectively, which guarantee that at most one stop is allocated at any station in a given route. Let $l(n_s)$ and $u(n_s)$ be the lower and upper bounds on the flow of arc $(n_s, d) \in \mathcal{A}(d)$, they control the minimum and maximum number of stops allocated at station n_s regardless of the bus corridor. It should also be noted that higher values on the bounds help with the connectivity of the BRT network and thus with the feasibility of the individual (e.g., $l(n_s) \geq 1$). The cost associated with flowing through arc $(n_s, d) \in \mathcal{A}(d)$ is zero. Finally, there is an arc between node d and node f that represents the total flow within the network. The cost associated with using this arc is also zero, and its lower and upper bounds are zero and $\sum_{r_k \in \mathcal{R}'} u(r_k)$, respectively.

Here, the objective is to minimize the total cost associated with the allocation of stops to stations, while satisfying flow capacity constraints. Moreover, since every node in the network is a transshipment node, the MCF problem can be seen as a circulation problem Ahuja et al. (1993).

6.4 Genetic operators and selection.

Since the routes allocated to each bus corridor are coded in a list of binary arrays, we can apply classical genetic operators like *bit flipping* mutation (on a randomly selected route) and *two-point crossover* Michalewicz (1996). It should be noted that applying a two-point crossover Michalewicz (1996) to a couple of routes r_1 and r_2 allocated to a given bus corridor c produces two routes r'_1 and r'_2. Figure 6 shows an example of two routes generated by applying the two-point crossover operator after the first and third positions of the binary arrays. Given a specific bus corridor, we apply the crossover operator as many times as the least number of routes present in one of the two parents. That is, assuming that parent p_1 has less routes allocated to corridor $\bar{c}$ than parent p_2, we select each route from p_1 and cross it with a randomly selected route from p_2. We apply the same procedure to every corridor.

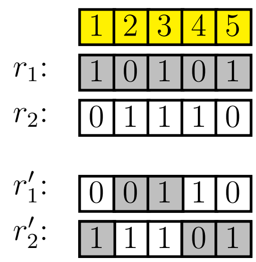


Figure 6: Example of the two-point crossover

Because of the routes' binary structure, the genetic operators are likely to produce individuals with the same fitness (clones), which is particularly critical in small-sized instances where such behavior may cause premature convergence to local optima. To avoid this problem, we use a tournament selection operator Miller and E. (1995) forbidding the selection of clones at each iteration.

7 Computational Experiments

The HGA was coded in Java Genetic Algorithm (JGA) Medaglia and Gutiérrez (2007), a publicly available Java-based object-oriented framework for solving optimization problems using evolutionary algorithms. JGA allows the user to focus on the specific application logic by reusing a set of built-in components. As shown in Figure 7, to implement the proposed algorithm in JGA, we extended the framework by coding the chromosome's genotype (`BRTRDPGenotype`), the fitness function evaluator (`BRTFitnessFunction`), the initial population procedure (`MCFInitialization`), the selection mechanism (`TournamentSelection`), and the crossover and mutation operators (`BRTRDPCrossover`, `BRTRDPFlipMutation`).

We solved both the model embedded in the fitness function evaluator and the MCF in the initial population generation procedure by connecting the corresponding Java component to the Xpress-MP optimizer, version 19.00.00.

To fine tune the HGA parameters, we conducted an experiment on a medium sized 21-station instance to explore the impact on solution quality of different levels of population

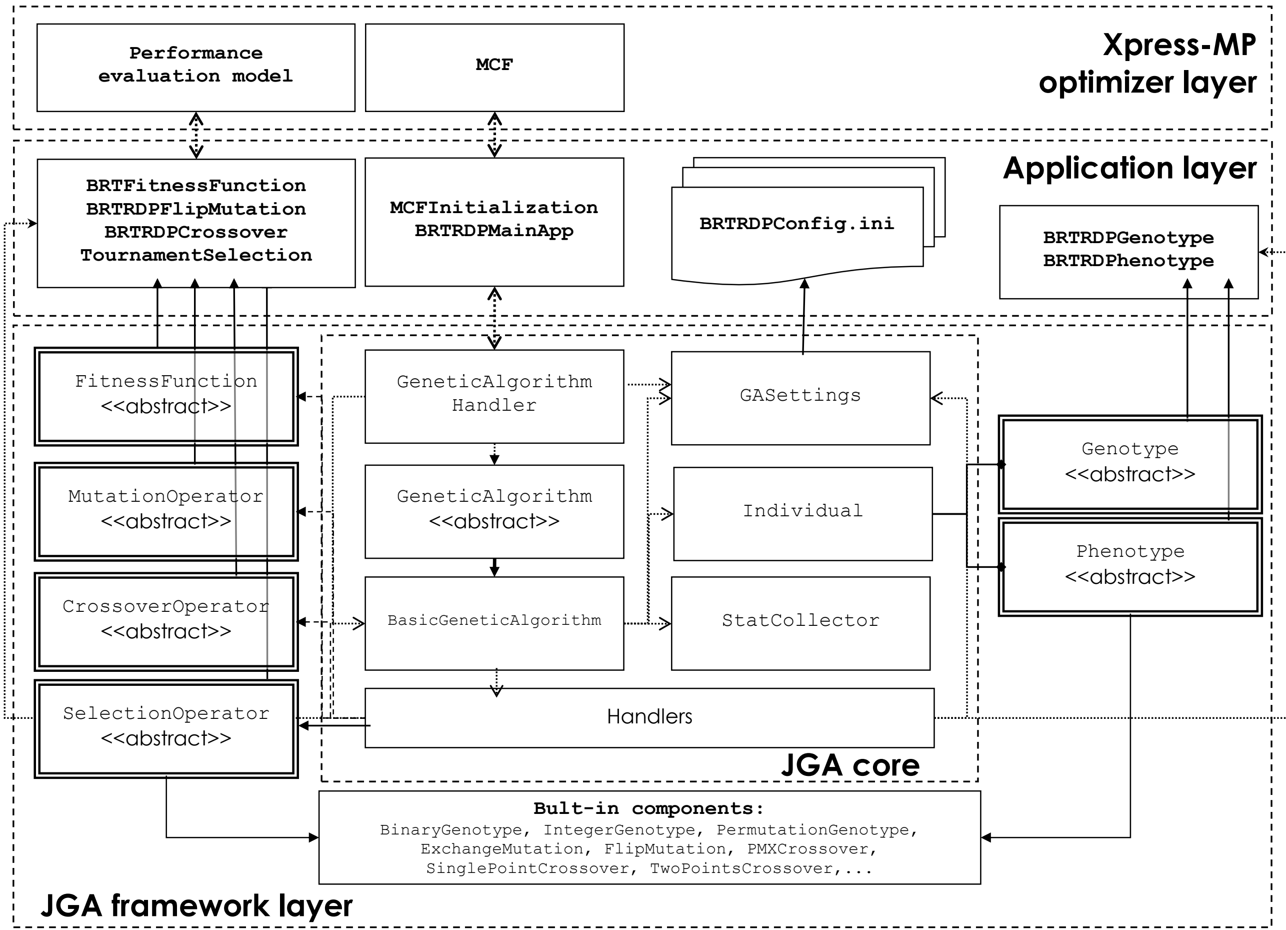


Figure 7: Implementation of the Genetic Algorithm in JGA

size ($P = 20, 30, 50$), maximum number of generations ($N = 100, 200, 500$), crossover rate ($p_c = 0.2, 0.3, 0.7$), and mutation rate ($p_m = 0.05, 0.08, 0.10$). Figure 8 reports the gap (in %) with respect to the best solution on each of the 81 ($= 3 \times 3 \times 3 \times 3$) parameter combinations. Based on these gaps, where the bright areas are preferred over the dark areas, we fixed $p_c = 0.3$ and $p_m = 0.1$ while selecting parameters P and N dependent on problem size. For the larger instances we fix $P = 30$ and $N = 200$. We conducted this experiment using JG^{2}A Bernal et al. (2009), an extension of JGA for computational grid environments; however, owing the diverse computational environment (one Dell OptiPlex 755 with an Intel Core Duo CPU running at 3.00GHz with 4 GB of RAM on Windows XP Professional; six Dell GX280-SD with an Intel Pentium IV CPU running at 2.5GHz with 512 MB of RAM on Windows XP Professional; and one Dell PowerEdge with an Intel Xeon Woodcrest 5120 processor with 4 GB of RAM on Windows Server 64 bits), we do not report the CPU time.

We tested the performance of the HGA by solving a set of 14 instances ranging from 5 to 40 stations. Instances 1 and 2 were adapted from two networks proposed by Marín (2007), representing a 6-station BRT system with 2 bus corridors (MAR-C2-S6) and a 9-station BRT system with 3 bus corridors (MAR-C3-S9). Instances 3 to 9 represent a BRT system with

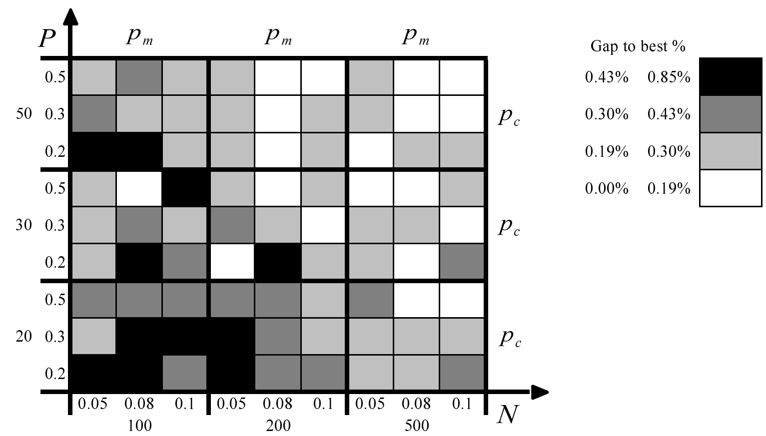
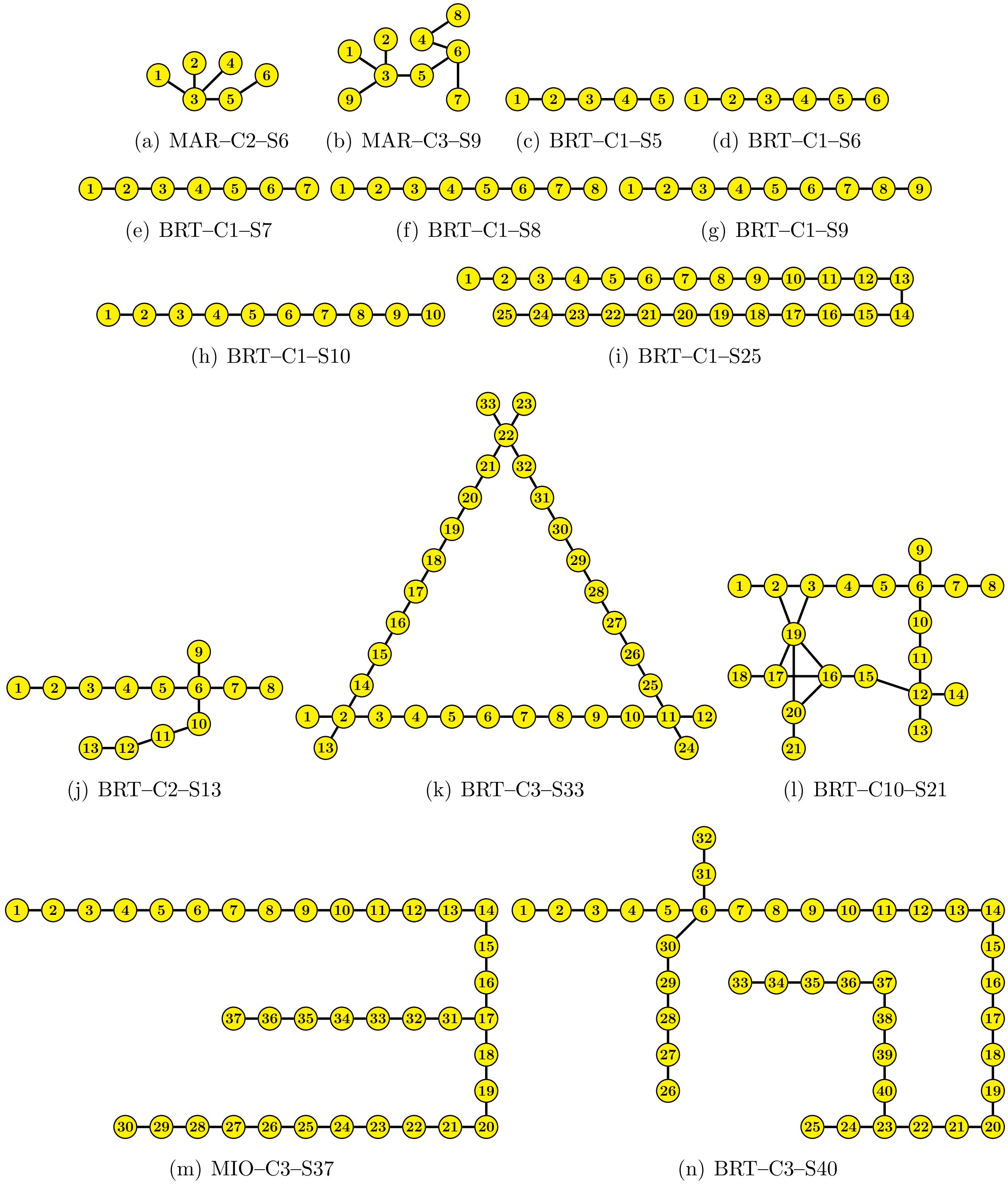


Figure 8: Effect of P, N, p_c, and p_m on the solution quality on a 21-station BRT system

one bus corridor and 5, 6, 7, 8, 9, 10, and 25 stations, labeled BRT-C1-S5, BRT-C1-S6, BRT-C1-S7, BRT-C1-S8, BRT-C1-S9, BRT-C1-S10, and BRT-C1-S25, respectively. Instance 10 represents a more complex 13-station BRT system with two bus corridors (BRT-C2-S13). Instance 11 is a triangular-shaped BRT system comprised by 33 stations and 3 bus corridors, labeled BRT-C3-S33. Instance 12 is a scaled representation of the TransMilenio BRT system with 10 bus corridors and 21 stations (BRT-C10-S21). Instance 13 labeled MIO-C3-S37, is a representation of the Mío, the BRT system of the city of Cali (one of the largest cities in Colombia); it has 37 stations and 3 of its most representative bus corridors. Finally, instance 14 is the largest instance with 3 corridors and 40 stations, labeled (BRT-C3-S40). The diversity of the topologies captured by this set of instances is shown in Figure 9. We performed the computational tests on a Dell Precision T7400 with an Intel Xeon CPU X5450 running at 3.00GHz with 8 GB of RAM on Windows Vista Ultimate.

For benchmark purposes, we calculated a dual bound based on the linear relaxation (LR) of the BRTRDP formulation proposed in §3 (1–9) using Xpress-MP 19.00.00 and CPLEX 12.1.0. For each instance, Table 1 reports the size of the underlying network in terms of the number of stations, bus corridors, and routes; as well as the problem size in terms of number of variables and constraints of the linear relaxation, its optimal value, and the elapsed time in seconds used by Xpress-MP and CPLEX to solve it. Because of the huge size of instances 9 to 14, neither Xpress-MP nor CPLEX could construct the linear relaxation problem. Notice the exponential number of variables and constraints as the number of stations and bus corridors increase.

For instances 1–12, we conducted 10 independent runs of the HGA, while for instances 13 and 14 we conducted 5 independent runs. Table 2 shows the HGA settings for the population size P and the maximum number of generations N, the best solution reported by the HGA, the average CPU time, and the gap between the best value of HGA and the linear relaxation (optimistic bound). For instances 1 through 6, the HGA consistently achieved the same solution in every run, while for instances 7, 8, 9, 10, 11, 12, 13, and 14 it found average gaps with respect to the best solution of 0.0016%, 0.0087%, 0.0139%, 0.0001%, 0.0061%, 0.0049%, 0.2083%, and 0.0094%, respectively. To check the overall quality of the best solution found by



(a) MAR–C2–S6 (b) MAR–C3–S9 (c) BRT–C1–S5 (d) BRT–C1–S6

(e) BRT–C1–S7 (f) BRT–C1–S8 (g) BRT–C1–S9

(h) BRT–C1–S10 (i) BRT–C1–S25

(j) BRT–C2–S13 (k) BRT–C3–S33 (l) BRT–C10–S21

(m) MIO–C3–S37 (n) BRT–C3–S40

Figure 9: Topologies of the BRT systems in the testbed

Table 1: Problem size and linear relaxation information for the 14 instances

Instance				Problem size				CPU time (s)	
No	Label	$\lvert\mathcal{S}\rvert$	$\lvert\mathcal{C}\rvert$	$\lvert\mathcal{R}\rvert$	Variables	Constraints	LR	Xpress-MP	CPLEX
1	MAR-C2-S6	6	2	15	10,410	1,351	240,264	0.01	0.02
2	MAR-C3-S9	9	3	34	113,108	7,498	553,762	0.70	0.40
3	BRT-C1-S5	5	1	26	5,012	1,726	752,380	0.01	0.01
4	BRT-C1-S6	6	1	57	19,014	6,077	863,576	0.05	0.08
5	BRT-C1-S7	7	1	120	64,248	19,580	925,392	0.71	0.47
6	BRT-C1-S8	8	1	247	200,414	59,131	988,631	2.58	2.84
7	BRT-C1-S9	9	1	502	589,676	169,978	1,062,204	15.65	20.73
8	BRT-C1-S10	10	1	1,013	1,659,286	470,009	1,119,764	177.56	273.88
9	BRT-C1-S25	25	1	$\approx 3 \times 10^7$	$\approx 4 \times 10^{18}$	$\approx 3 \times 10^{11}$		–	–
10	BRT-C2-S13	13	2	304	$\approx 2 \times 10^7$	$\approx 3 \times 10^6$	–	–	–
11	BRT-C3-S33*	33	3	12,249	$\approx 2 \times 10^7$	$\approx 2 \times 10^7$	–	–	–
12	BRT-C10-S21*	21	10	2,701	$\approx 3 \times 10^8$	$\approx 9 \times 10^5$	–	–	–
13	MIO-C3-S37*	37	3	$\approx 1 \times 10^9$	$\approx 7 \times 10^{21}$	$\approx 1 \times 10^{13}$	–	–	–
14	BRT-C3-S40*	40	3	$\approx 3 \times 10^7$	$\approx 4 \times 10^{18}$	$\approx 1 \times 10^{11}$	–	–	–

* The number of variables and constraints were calculated based on the size of each bus corridor.

HGA, we compared it against the dual bound obtained by the linear relaxation. For instances 1 through 8, the average gap with respect to the linear relaxation was 0.10%, and never greater than 0.20%. Even though those instances may be considered small, the gaps of less than 1% provide some evidence that this good performance could scale to larger instances, for which no good bounds are known. In addition, in terms of solution time, the HGA has also proven robust. We found that the average coefficient of variation in the time spent by the algorithm was 0.0567, meaning that, regardless of the random stream, the elapsed time is consistently the same.

We finally tested whether the solutions found were indeed optimal. Therefore, we directly solved the MIP of the BRTRDP formulation proposed in §3 (1–9) for instances 1 to 8 (i.e., the instances for which the optimizers were able to solve the linear relaxation) using Xpress-MP 19.00.00 and CPLEX 12.1.0. We set the maximum running time of the optimizers to 12 hours, almost 160 times the time spent by the HGA to solve the largest of the 8 instances. As Table 3 shows, Xpress-MP and CPLEX both reached the optimal solution for instances 1, 2, 3, and 4 in less than 80 seconds. Remarkably, HGA also obtained the same optimal solution in every run within a reasonable time (ranging from 2.778 to 29.281 seconds on average). However, for the remaining instances, none of the optimizers could find the optimal solution within the time limit. For instances 5 and 6, the HGA was able to find the same solution found by the optimizers, but we cannot guarantee that the solution is optimal since the optimizers were not able to close the gap. For instances 7 and 8, the HGA was 0.016% and 0.431%, better, respectively, than the best solution found by Xpress-MP. For those instances, CPLEX found no integer solution within the allowed time limit.

Finally, Figure 10 shows a graphical representation of the solutions found by the HGA for instances BRT-C1-S10, BRT-C2S13, and MIO-C3-S37, including the stops and the number of buses allocated to serve each route. Additionally, Table 4 presents the OD matrix for the BRT-C1-S10 system, where the high-demand (o, d) pairs are in bold.

Table 2: Performance of HGA and comparison with the Linear Relaxation

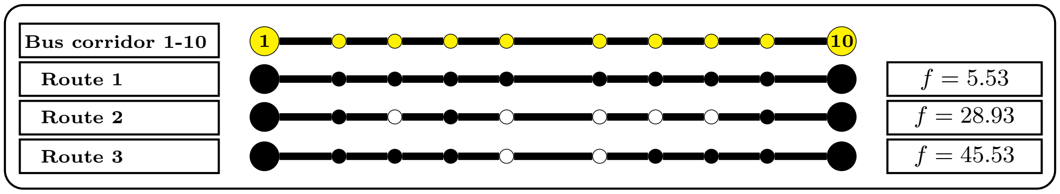
Instance	P	N	HGA best	Average time (s)	Gap (LR-HGA best) %
1	5	100	240,353	2.78	0.03%
2	10	100	554,154	16.70	0.07%
3	10	200	752,672	8.42	0.04%
4	20	200	864,346	29.28	0.09%
5	30	200	926,371	72.56	0.11%
6	30	200	989,771	122.84	0.12%
7	30	200	1,063,991	193.79	0.17%
8	30	200	1,121,948	276.85	0.20%
9	30	200	93,394,080	22,068.05	–
10	30	200	58,904,536	596.60	–
11	30	200	215,677,088	25,360.37	–
12	30	200	183,861,728	17,561.27	–
13	30	200	405,040,576	146,729.84	–
14	30	200	347,142,432	79,297.08	–

Table 3: Performance of HGA against the MIP optimizers Xpress-MP and CPLEX with the time limit of 43,200 seconds

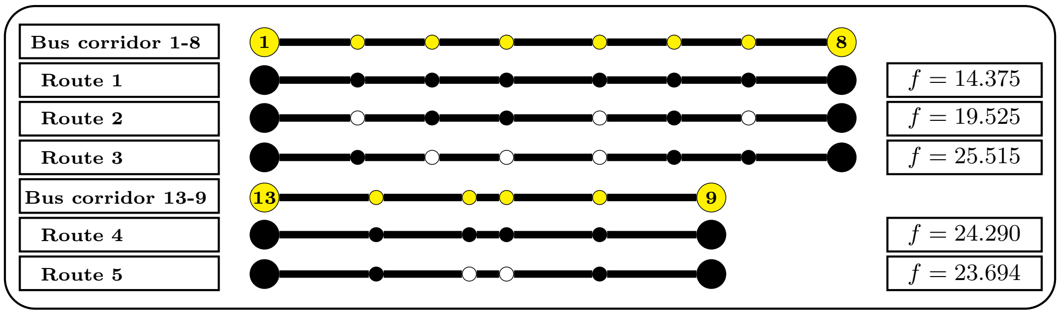
	Xpress-MP		Gap (HGA best-	CPLEX		Gap (HGA best-
Instance	Best integer	Time (s)	Xpress-Mp) %	Best integer	Time (s)	CPLEX) %
1	240,353	0.31	0.000%	240,335	0.22	0.000%
2	554,154	72.16	0.000%	554,154	25.43	0.000%
3	752,672	0.42	0.000%	752,672	0.33	0.000%
4	864,346	7.33	0.000%	864,346	8.47	0.000%
5	926,371	43,200.00	0.000%	926,371	43,200.00	0.000%
6	989,779	43,200.00	0.000%	989,771	43,200.00	0.000%
7	1,064,159	43,200.00	0.016%	–	43,200.00	–
8	1,126,784	43,200.00	0.431%	–	43,200.00	–

Table 4: OD matrix for the BRT-C1-S10 system

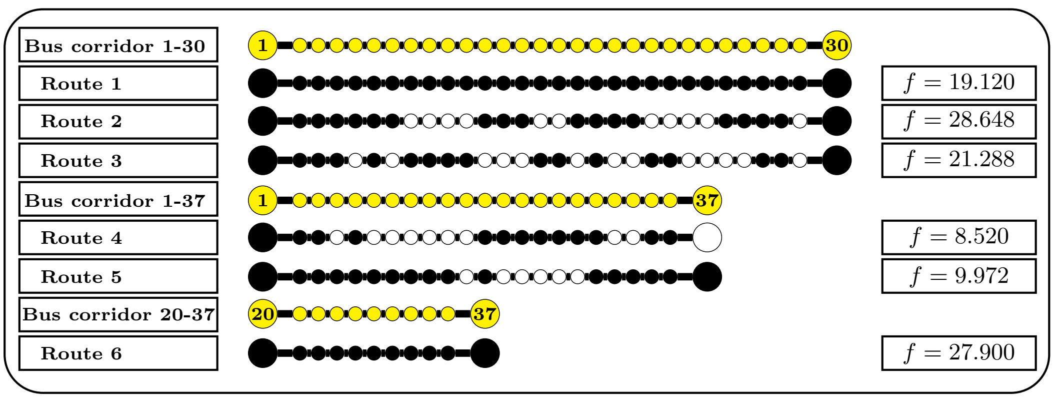
OD	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
1	0	2	5	27	3	3	11	7	2	1
2	3	0	2	24	18	5	58	21	26	0
3	5	3	0	5	8	6	62	19	15	3
4	43	**149**	20	0	0	10	**256**	**196**	**264**	57
5	3	17	3	8	0	0	10	2	14	0
6	4	9	8	17	0	0	1	5	3	10
7	26	106	79	**227**	5	2	0	0	18	8
8	5	36	26	**151**	7	0	4	0	0	0
9	7	19	23	**233**	21	4	8	0	0	1
10	1	3	4	49	12	4	15	2	1	0



(a) BRT–C1–S10



(b) BRT–C2–S13



(c) MIO–C3–S37

Figure 10: Solutions found by the HGA

For the case of the BRT-C1-S10 system, note that the segments without stops in routes 2 and 3 significantly benefit passengers willing to travel between the stations with high demands. For example, the 497 passengers traveling between stations 4 and 9 (i.e., 264 from station 4 to station 9 and 233 from station 9 to station 4) can expedite their travel without intermediate stops by using route 2. A similar situation occurs between stations 4 and 7 on route 3. On the other hand, route 1 stops at every station to cover lower demand o-d pairs. This expected behavior has been observed in other BRT systems with different topologies. Regardless of the instance, note that there is always a route that stops at every station in each bus corridor; and, when there are multiple routes allocated to a bus corridor, they try to cover different stations, complementing each other. For instance, see routes 4 and 5 of the bus corridor 1-37 in the solution for the MIO-C3-S37.

In terms of frequencies, for the case of the MIO-C3-S37 system, the number of buses allocated to the 6 proposed routes were 19.12, 28.65, 21.29, 8.52, 9.97, and 27.90, respectively;

using about 116 out of the 150 available buses. Furthermore, since the OD matrix was built using a three-hour peak in the morning, buses serving those routes should be dispatched roughly every 9, 6, 8, 21, 18, and 6 minutes, respectively.

8 Concluding Remarks

This study was motivated by the ever-increasing interest of city planners in adopting BRT systems as competitive alternatives to rail-based systems. Owing to the vast success of systems like TransMilenio in Bogotá, many cities have been rapidly adopting this type of public transit system. However, eventually, BRT system operators face the problem of (re)designing the bus routes to increase system efficiency and overall user satisfaction.

To address this problem, we formulate the Bus Rapid Transit Route Design Problem (BRTRDP) as a MIP with an underlying network structure. However, because this problem's direct solution is out of reach for practical size problems, we propose a decomposition strategy that, given a set of routes, decouples the route selection decisions from the BRT system performance evaluation. This evaluation is solved using a large-scale linear programming technique that reduces the computational time needed to evaluate the performance of any given solution. To illustrate the decomposition scheme, we present a hybrid genetic algorithm (HGA) in which each solution is encoded in a binary genotype with multiple fragments, representing a set of routes able to operate the BRT system.

To test the HGA's performance, we solved 14 instances: two adapted from the literature and 12 emulating realistic BRT systems. Two of the largest instances are a 21-station scaled version of the widely popular TransMilenio BRT system and a 37-station built from 3 representative bus corridors of the Mío BRT system of Cali (Colombia). The HGA reached very accurate solutions for all instances, and in 8 out of 14 problems obtained a provably optimal solution within less than 0.2% of a strong dual bound. In 4 out of 14 instances, the HGA obtained the optimal solution.

We believe that, beyond its genetic search features, this application of the HGA has shown that the proposed decomposition strategy can successfully tackle real-sized instances of the BRTRDP. Moreover, the same decomposition strategy could easily be embedded into another metaheuristic framework such as a tabu search or variable neighborhood search.

Research is currently underway to improve the execution time of the fitness evaluation procedure by implementing a specialized shortest path algorithm.

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